Human Resource Allocation in Project Management
- Management Science Approach -

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Abstract

The applications of management science to project management has long been limited in scope and depth. It can simply be said that the scheduling for planning and executing projects is based upon PERT method followed by integer programming approach to human resource allocation for minimizing a total cost subject to the constraints regarding allocation such as integer (1/0) for presenting whether or not a task is allocated.

In this paper, presented is the result of in-depth analyses of the problem accounting for project specific characteristics such as allocation of workers with different levels of skills to a certain task which is involved in project tasks consisting of a series of tasks with simultaneous accomplishment. Based upon the logistic equation representing the dynamic working process, the results show the improvement of economic performance and management decision for overcoming chaotic phenomena observed during carrying out a series of works in projects.

Keywords: resource allocation, scheduling, project management, analytical solution method

1. Introduction

Much attention has been paid in recent years to project management in Japan. Our department was established as the first academia specialized for project management in 1997. Japanese Project Management Forum was funded with over 1,000 members for promoting the project management professionalism in Japan under the leadership of the Ministry of Economy, Trade and Industry. Also PMI Tokyo, Japan Chapter was established for promoting the PMP certification under the leadership of the Ministry of Land, Infrastructure and Transport. In 1998, the Society of Project Management was organized as a first academic society for project management, and has about 900 members.

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Table 1 Contract Ranking of Engineering and Construction Firms

Middle East Area	South Asia Area
1. Mitsubishi Heavy Industries, Ltd.	1. Mitsubishi Heavy Industries, Ltd.
2. Chiyoda Corporation	2. Tnafalgar House Engineering & Construction
3. TECHNIP	3. JGC Corporation
4. Consolidated Contractors Intl. Co. SAL	4. Toyo Engineering Corporation
5. Bechtel Group Inc.	5. Bilfinger + Berger Bau AG
6. Ansaldo SPA	6. Hochtief AG
7. Ballast Nedam Construction Intl. BV	7. Kumagai Gumi Co., Ltd.
8. Trafalgar House Engineering & Construction	8. Shimizu Corporation
9. ABB SAE Sadelmi SPA	9. Fluor Daniel Inc.
10. Daewoo Corp. Engineering & Construction	10. Nishimatsu Construction Co., Ltd.

Engineering News-Record, Aug.28'95, The McGraw-Hill Companies Inc.
□ shows Japanese firm.

Though project management should be performed by the standardized process, some degrees of strength for the project execution at Japanese engineering firms could be confirmed by the share of construction market in 80's and early 90's as shown in Table 1. It is, however, very difficult to analyze and describe their strength, because all engineering activities are based on work performance done by white collar workers. Because of the restructuring of market structure and recession of Japanese economy, it is requested for those firms to devise the ways of increasing work performance of projects in order to compete in the global market. One of the key factors for successful projects in global competition is how to be excellent in human resource allocation in all projects.

In this paper, the authors take up this problem, and describe a resource allocation model and problem-solving approaches which reflect actual project execution.

2. Brief Review

The traditional management science approach for the project management is shown in Fig. 1. In this approach, a start of this information flow is requirements and they are broken down into the deliverables, tasks, costs and schedule.

Four major reasons can be stated for the inappropriateness of traditional approach.

(1) Though the project management assumes dynamic interactions among stakeholders, the approach statically describes taken processes, required resources and deliverables.

(2) Though the project environment becomes critical to reduce implicitly incorporated operational

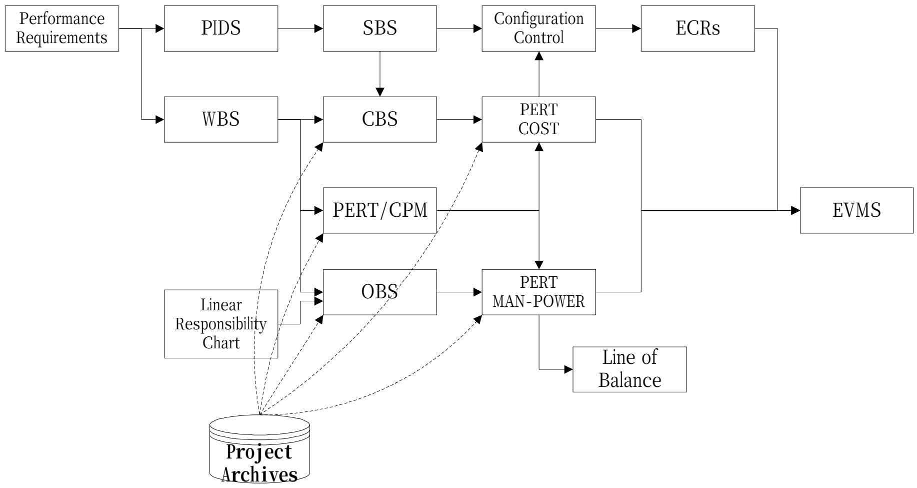

Fig. 1 Operational Relation of Project Management Methods

margins, the approach is based on information which explicitly and rigorously specified at the beginning of the project.

(3) Though the approach requests detail information, the resulting complexity and schedule does not allow agile and reliable analysis and real-time feedback.

(4) Though all tasks are performed by human engineers and clerks, the approach disregards human factors.

In order to cope with the above mentioned problems, more human resources are requested in the project management. Consequently, numerous papers have been published for solving the RCPSP (resource-constrained project scheduling problem) with realistic conditions.[1,2,3] These approaches are, however, too theoretical to realize actual project management.

Invalidation of these assumptions, and consequently of these techniques, creates demand for new management paradigms and technologies.

3. New Approach to Human Resource Management in Project Management

The work breakdown structure (WBS) is the key part of the project work plan. It defines the work to be performed, specifies the required engineering type and level, and settles a base for controlling project schedule and responsibility. For consulting and engineering firms, the matrix organizations provide efficient project execution environment with emphasis on the functionality of each discipline.

Under the matrix organization, all project tasks in WBS are assigned to each discipline through work packages. In the discipline, tasks assigned in the work package are shared among engineers by each discipline manager. For the discipline manager, the success of project in the matrix organizations depends on a balance between managing discipline resources and providing functional experts. Therefore most important role of the discipline manager is to find out a certain degrees of freedom in the work packages for keeping his freehand against the future projects, though the work package restrictedly specifies its scope, schedule, deliverables and work volumes.

Based on the authors' own experience in an engineering firm and deep observation of the behavior of discipline managers, following two strategic approaches may be applied by the discipline managers.

1. Relocation of a specified work volume based on engineer's skill level.
   Considering required skill levels for performing work, a discipline manager assigns multiple engineers who have different level of skill to share same work volume.

2. Relocation of a specified work schedule based on engineer's skill level.
   Considering specified deadline of the deliverables, a discipline manager assigns multiple engineers to share the same work schedule.

Consequently, in order to compete in a recent severe competition, project management problem should be solved as a resource allocation problem in a discipline.

3.1 Static Approach for the Human Resource Allocation

Tasks associated with its work volume is specified in the work package. After each task is divided into several sub-tasks to meet available human resources in the discipline, the progress of the task is measured by man-hours consumed. Therefore, the following assumptions are made for resource allocation problem.

1) A task in the work package can be divided into a multiple of graded sub-tasks, and work volumes are specified to the graded sub-tasks.
2) Discipline engineers can be classified into engineer's classes based on their engineering experiences and capabilities.
3) The work efficiencies of the graded sub-task can be defined for corresponding engineer's classes, respectively. There exist differences of work efficiency between adequate engineer's class and inadequate engineer's class against the assigned sub-task.
4) When an engineer charges some parts of work to the lower grade of inadequate sub-task, the

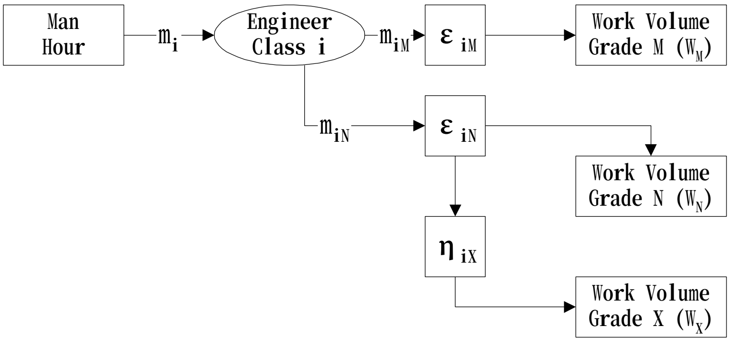

Fig. 2 Task Execution Model for a Discipline Engineer

engineer should perform additional tasks, such as administrative work and/or clerical work.

Taking the above assumptions into consideration, a task execution model for discipline engineers can be described as shown in Fig. 2. In this model, we make a number of actual and simplifying assumptions:

1) Man-hour of the class (i) engineer is allocated to three grades of sub-tasks.
2) The class (i) engineer is an adequate for the grade (M) of sub-task and charged for the lower grade (N) of sub-task.
3) The engineer additionally has to handle the grade (X) of sub-task associated with the grade (M) of sub-task.

The relation between the work volume divided into the graded sub-task and the man-hours for executing the graded sub-task is represented in the following equations:

$$m_i = m_{iM} + m_{iN} \tag{1}$$
$$e_{iM} \cdot m_{iM} = W_M \tag{2}$$
$$e_{iN} \cdot m_{iN} = W_N \tag{3}$$
$$e_{iN} \cdot h_{iX} \cdot m_{iN} = W_X \tag{4}$$

where

m_k: Man-hours for the class (k) engineer

m_{kL}: Man-hours comsumed for the grade L task by the class (k) engineer

W_L: Work volume of the grade L task

e_{kL}: Work efficiency of the class (k) engineer when he performs the grade L task

h_{iL}: Compensation factor of work efficiency for the class (k) engineer when he performs the grade L task as a side-work

Considering the organizational operation in the discipline, the following assumptions are added to describe the resource allocation problem in the discipline.

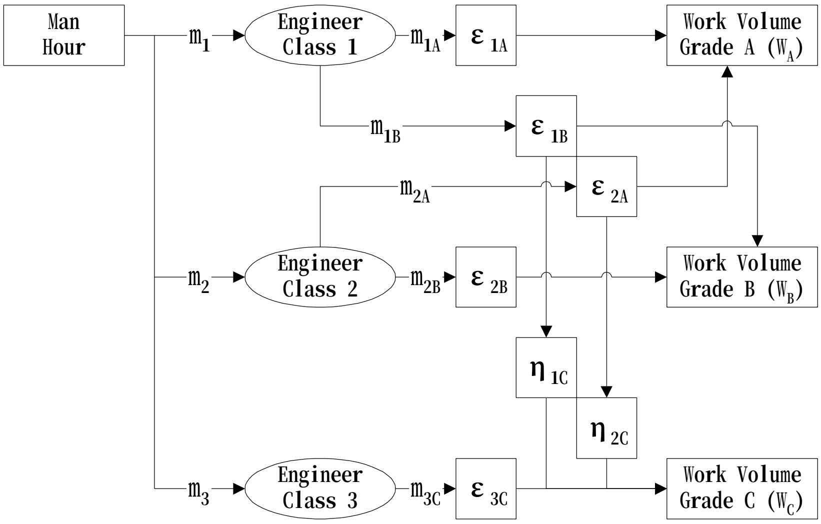

Fig. 3 Mon-hour Allocation Model

1) All tasks are graded into the three grade, such as A, B and C, and all engineers are also classified into three class according to the task grade.

First class engineer: He / She is assigned as a chief discipline engineer from the project and is in charge of the grade A of tasks, such as approving task of the deliverables and correspondences.

Second class engineer: He / She is a real workforce and is assigned as a lead engineer from the discipline manager and is in charge of the grade B of tasks, such as engineering work. He / She also has a capability to act as a first class engineer.

Third class engineer: He / She is a supporting staff and is in charge of the grade C of tasks, such as engineering support-work and engineering administration.

2) The first and second class of engineers are complementary to each other for the grade A and B tasks. On the other hand, the third engineer cannot be assigned to the grade A and B tasks.

Based on the above mentioned assumptions, the resource allocation problem in the discipline is summarized as a man-hour allocation model shown in Fig. 3.

The resource allocation problem is described by:

Minimize $$\mathrm{Z} = c\sum_{i=1}^{3} m_i \quad (5)$$

Subject to: (6)

$$m_1 = m_{1A} + m_{1B} \quad (7)$$

$$m_2 = m_{2A} + m_{2B} \quad (8)$$

$$m_3 = m_{3C} \quad (9)$$

Table 1 Sets of Combination of Basic Variables

Soln. No.	m_{1A}	m_{1B}	m_{2A}	m_{2B}	m_{3C}	λ
1	1	0	1	1	0	0
2	1	1	0	1	0	0
3	1	0	0	1	1	0
4	1	1	1	0	0	0
5	1	1	0	0	0	1
6	1	1	0	0	1	0
7	0	1	1	1	0	0
8	0	0	1	1	0	1
9	0	0	1	1	1	0
10	0	1	1	0	0	1
11	0	1	1	0	1	0

Table 2 Sets of Basic Feasible Solutions with its Feasible Region

Soln .No.	Feasible Region	$m_{1A}, m_{1B}, m_{2A}, m_{2B}, m_{3C}$
1	$0 \leqq W_C \leqq \eta_{2C} W_A$	$(W_A - \eta_{2C} W_C)/\varepsilon_{1A}$, 0, $W_C/\varepsilon_{2A}\eta_{2C}$, W_B/ε_{2B}, 0
2	$0 \leqq W_C \leqq \eta_{1C} W_B$	W_A/ε_{1A}, $W_C/\varepsilon_{1B}\eta_{1C}$, 0, $(W_B - \eta_{1C} W_C)/\varepsilon_{2B}$, 0
3	$0 \leqq W_C$	W_A/ε_{1A}, 0, 0, W_B/ε_{2B}, W_C/ε_{3C}
4	$\eta_{1C} W_B \leqq W_C \leqq \eta_{2C} W_A$	$(W_A + \eta_{2C} W_B/\eta_{1C} - \eta_{2C} W_C)/\varepsilon_{1A}$, W_B/ε_{1B}, $(W_C - \eta_{2C} W_B)/\varepsilon_{2A}\eta_{2C}$, 0, 0
5	$W_C \leqq \eta_{1C} W_B$	W_A/ε_{1A}, W_B/ε_{1B}, 0, 0, 0
6	$\eta_{1C} W_B \leqq W_C$	W_A/ε_{1A}, W_B/ε_{1B}, 0, 0, $(W_C - \eta_{1C} W_B)/\varepsilon_{3C}$
7	$\eta_{2C} W_A \leqq W_C \leqq \eta_{2C} W_A + \eta_{1C} W_B$	0, $(W_C - \eta_{2C} W_A)/\varepsilon_{1B}\eta_{1C}$, W_A/ε_{2A}, $(\eta_{2C} W_A + \eta_{1C} W_B - W_C)/\varepsilon_{1A}\eta_{1C}$, 0
8	$W_C \leqq \eta_{2C} W_A$	0, 0, W_A/ε_{2A}, W_B/ε_{1B}, 0
9	$\eta_{2C} W_A \leqq W_C$	0, 0, W_A/ε_{2A}, W_B/ε_{1B}, $(W_C - \eta_{2C} W_A)/\varepsilon_{3C}$
10	$W_C \leqq \eta_{2C} W_A + \eta_{1C} W_B$	0, W_B/ε_{1B}, W_A/ε_{2A}, 0, 0
11	$\eta_{2C} W_A + \eta_{1C} W_B \leqq W_C$	0, W_B/ε_{1B}, W_A/ε_{2A}, 0, $(W_C - \eta_{2C} W_A - \eta_{1C} W_B)/\varepsilon_{3C}$

$$e_{1A} \cdot m_{1A} + e_{2A} \cdot m_{2A} = W_A \quad (10)$$

$$e_{1B} \cdot m_{1B} + e_{2B} \cdot m_{2B} = W_B \quad (11)$$

$$e_{1B} \cdot h_{1C} \cdot m_{1B} + e_{2A} \cdot h_{2C} \cdot m_{2A} + e_{3C} \cdot m_{3C} \geq W_C \quad (12)$$

The approach proposed herein centers on a problem with the Linear Programming (LP) form. The problem is solved by applying an analytical solution method for the LP problem which enables possible essential understanding of the solution space for the problem.[5] The method applied here is summarized in the Appendix 1 for your convenience. The analysis of optimal solution space derived by the analytical solution method gives the following three results which cannot be obtained by a numerical approach:

(1) Infeasible combination of basic variables

(2) Combination of variables which cannot be optimal

(3) Optimal allocation and condition for human resource allocation problem

In accordance with the analytical solution method, the set of combinations of feasible solutions is

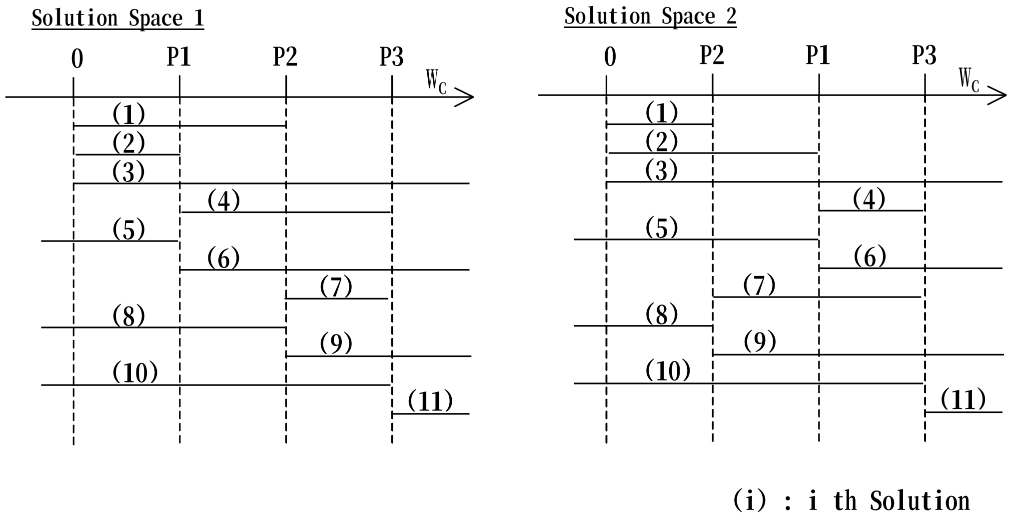

Fig. 4 Solution Space for Man-hours Allocation Problem

shown in Table 1 and feasible regions and optimal solutions are shown in Table 2. There are 11 feasible solutions.

The solution number 3 shows an ideal assignment where each class of engineers has only charge of their own task. The optimal region of each feasible solutions is depend on the following three transition points, e and h.

$$\mathrm{P1} = h_{1C} W_B \tag{13}$$

$$\mathrm{P2} = h_{2C} W_A \tag{14}$$

$$\mathrm{P3} = h_{2C} W_A + h_{1C} W_B \tag{15}$$

P1, P2 and P3 define two solution spaces as shown in Fig. 4, where each feasible solution is mapped in its feasible region. From the solution space 1 shown in Fig. 4, it is understood that the solution number 1, 3, 4, 6, 8 and 10 are feasible solutions in the range of P1 < W_C < P2.

Considering P1 < P2 < P3 as a practical condition, the optimal solution space is defined as shown in Fig. 5. Using this diagram, it is easily recognized that the optimal solution with optimality conditions and its sensitivity in the whole solution space. For example, if the solution number 5 is the optimal solution in the range of W_C < P1 and the work volume of grade C is changed from W_C < P1 to P3 < W_C under the following condition, the optimal solution is transferred from No. 5 to No. 4 at the point P1, and No.4 is transferred to No. 11 at the point P3.

$$\frac{1}{e_{2A} h_{2C}} - \frac{1}{e_{1A} h_{2C}} < \frac{1}{e_{3C}} \tag{16}$$

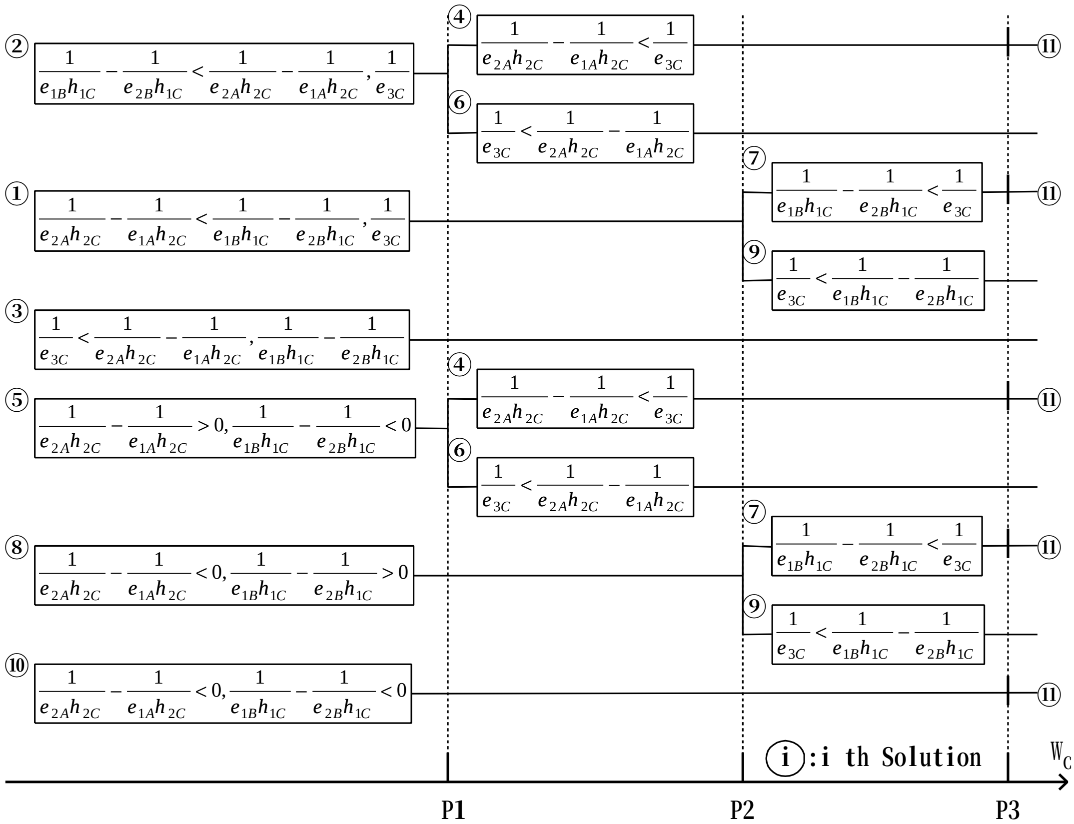

Fig. 5 Optimal Solution Space for Man-hours Allocation Problem

(Case: P1 < P2 < P3)

In order to derive heuristics in the resource allocation problem, the following conditions are introduced to practically reflect performance differences among each engineer's class.

$$e_{1A} > e_{2B} > e_{1B} > e_{2A}$$

$$\frac{1}{e_{2A}} - \frac{1}{e_{1A}} < \frac{1}{e_{2B}} - \frac{1}{e_{1B}}$$

$$h_{1C} > h_{2C}$$

$$e_{3C} > e_{1B}h_{1C} > e_{2A}h_{2C}$$

These conditions are summarized as the following equation.

$$\frac{1}{e_{2A}h_{2C}} - \frac{1}{e_{1A}h_{2C}} < \frac{1}{e_{1B}h_{1C}} - \frac{1}{e_{2B}h_{1C}} < \frac{1}{e_{3C}} \tag{17}$$

Applying the above condition, the solution No. 2, 4 or 11 controls the optimal solution space shown in Fig. 5 and the following heuristics between the work volumes and the man-hours can be derived.

Heuristic rule 1: *If* the work volume of grade C is less than P1,

then the class 1 engineer should perform the grade A task and share the grade

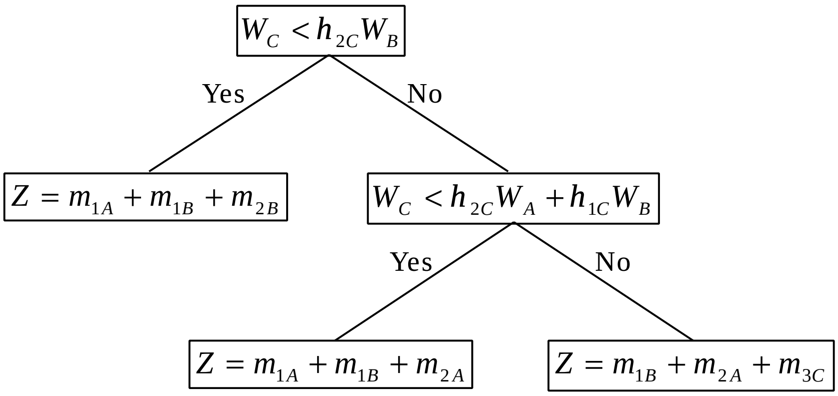

Fig. 6 A Decision Tree for Human Resource Allocation Problem in Project Management

B task with the class 2 engineer and
the class 2 engineer should concentrate on the grade B task and
the grade C may be performed as the side-work of the class 1 engineer.

Heuristic rule 2: *If* the work volume of grade C is greater than P1 and less than P2,
then the class 1 engineer should perform the grade A task and the grade B task and
the class 2 engineer should support the class 1 engineer and share the grade A task and
the grade C task may be performed as the side-work of the class 1 and the class 2 engineers.

Heuristic rule 3: *If* the work volume of grade C is greater than P2,
then the class 1 engineer should support the class 2 engineer to follow the grade B task and
the class 2 engineer should act as the chief engineer to perform the grade A task and
the grade C task should be performed by the class 3 engineer under the support of the class 1 and 2 engineers.

As the summary of the above heuristics, a decision tree is presented in Fig. 6.

3.2 Dynamic Approach for the Human Resource Allocation

In the previous section, the task is statically divided into multiple sub-tasks in accordance with the associated engineer's levels. In the execution phase, the discipline manager may contrive to reduce a work load of the higher class of engineers and to assign lower class of engineers as long as possible, because the higher class of engineers are the scarce resource for the discipline. Both time

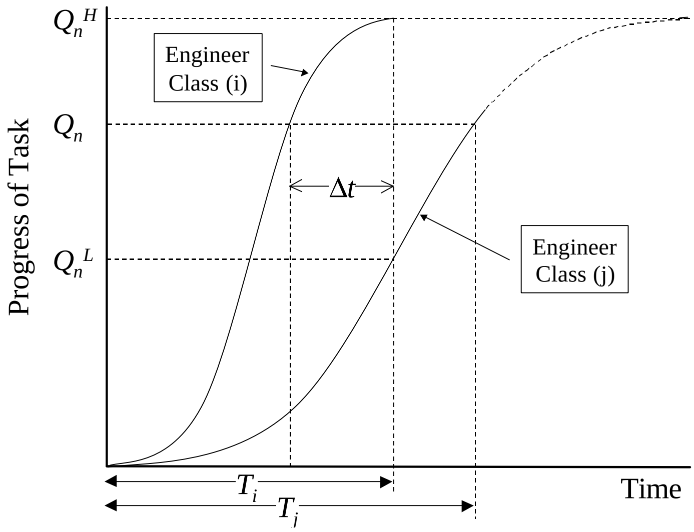

Fig. 7 Work Performance Curve

and cost incurred by processing a sub-task depend on the class of engineers assigned. In a practical situation, the lower class of engineer may be charged at the beginning and the higher class of engineer take over the task. Therefore the following assumption may be added dynamically to divide the task to reduce cost and shorten schedule by switching engineers.

1) Progress or cumulative effort of a engineering task can be defined as an S-shaped curve as shown in Fig. 7.[4] In Fig. 7, a class (i) engineers has a higher performance than class (j) engineers, and the class (i) engineer is more expensive than the class (j) engineer.
2) The class (i) engineer takes over the class (j) engineer's task, after the class (j) engineer performs the task up to a certain progress, Q_n.
3) The class (j) engineer has to keep the original schedule to perform the task from zero to progress, Q_n. The class (i) engineer does not exactly follow the precedence order in the original schedule and may handle remaining task from Q_n to Q_n^H by the next milestone on the time chart.

Concept of the above assumptions 1 and 2 is shown in Fig. 8. When the class (j) engineer supports the class (i) engineer, total cost of the task may be reduced under the following condition.

$$\frac{T_i - \Delta t}{T_j} > \left(\frac{C_j}{C_i}\right) \cdot \left(\frac{N_j}{N_i}\right) \tag{18}$$

where

$$T_i = f_i^{-1}(Q_n^H) \tag{18-1}$$

$$T_j = f_j^{-1}(Q_n) \tag{18-2}$$

$$\Delta t = f_i^{-1}(Q_n^H) - f_i^{-1}(Q_n) \tag{18-3}$$

C_i, C_j: Unit cost of the class (i) engineers and the class (j) engineers, where $C_i > C_j$.

N_i, N_j: Number of the class (i) engineers and the class (j) engineers

f_i, f_j,: Efficiency function of the class (i) engineers and the class (j) engineers

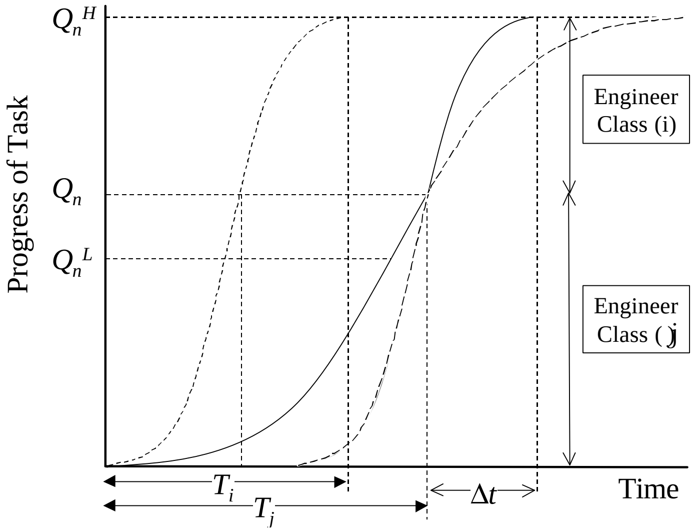

Fig. 8 Take over of a Task between Two Engineer Classes

The condition Eq.(18) defines a constraint to specify the take-over point Q_n. In the actual situation, the number of engineer's ratio is $(N_j/N_i) \geqq 1$ and their cost ratio is $(C_j/C_i)<1$. Therefore the load of the class (i) engineer is reduced from T_i to Δt and the total man-hour cost is also reduced.

In order to meet requested tight schedule, the discipline manager may be forced to assign the class (i) engineers at the tasks on the critical path. As the above mentioned approach requires additional time ($T_j + \Delta t - T_i$) for the planned schedule, it is necessary to take another strategic approach based on the assumption 3. The total duration of the project is defined below.

$$T_c = \sum_{nc=1}^{NC} T_{j,nc} + \sum_{nc=1}^{NC} \Delta t_{nc} \tag{19}$$

where *nc* shows a task on the critical path and *NC* is the total number of the task on the critical path. If some tasks collected in $\Sigma \Delta t_{nc}$ have a chance to shift to parallel tasks, the resulting schedule may shorten. Accordingly, the rescheduling procedure is proposed as follows.

Step 1: Select as many as possible tasks on the critical path which can be shared by multiple class of engineers and have large Δt_{nc}.

Step 2: Layout the selected and separated tasks somewhere between the original task and the successor milestone where the discipline formally delivers its deliverables.

Step 3: If the critical path is changed by the above rearrangement, repeat Step 1 and 2.

4. Concluding Remarks

As the result of analysis in project management, it is expected the work performance which leads to cost reduction for a specified schedule. Schedule control in project execution has been done traditionally by using the PERT/CPM without careful work assignment. The above mentioned results suggest the necessity of more careful analysis on work assignment before using the PERT/CPM, for the competitive project executions.

The authors have investigated more realistic problems based on the real world project executions. The problem of human resource allocation is one of these problems and the analyses by management science approach described in this study have given us the feeling that there are some rooms for the related research to be investigated to improve the traditional project management methods. The more in-depth-analysis of problems with modeling will be necessary in general. There is no exception for project management.

At present, the authors are directed towards the further analysis of actual project activities based on the behavior science. As the preparatory study, a fundamental understanding of schedule delay is discussed based on a engineer's behavior model for project task. A detailed discussion on this matter will be given elsewhere.

Acknowledgment

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Appendix: Analytical Solution Method for LP

A. Theoretical Basis

The linear programming problem may be stated as follows using matrix notation.

Maximize

$$P = c^T x \tag{A1}$$

Subject to

$$Ax = b \tag{A2}$$

$$x \geq 0 \tag{A3}$$

where variable *x* include so-called "slack variables" introduced when inequality constraints are made into equality constraints. *A* is an m x n matrix with n > m in Eq.A.3, where n is the number of variables and m is the number of constraints. Now consider solving m simultaneous Eq.A3. By partitioning *x* into basic variables x_B and nonbasic variables x_N and *A* into sub-matrices *B* and *N*, we can get

$$A = (B\,N) \tag{A4}$$

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} \tag{A5}$$

$$Ax = Bx_B + Nx_N = b \tag{A6}$$

$$x_B = B^{-1}b - B^{-1}Nx_N \tag{A7}$$

Also by partitioning the cost vector *c* into c_B and c_N, the objective function *P* can be written as follows.

$$\begin{aligned} P &= c^T x = \left(c_B^T\, c_N^T\right)\begin{pmatrix} x_B \\ x_N \end{pmatrix} \\ &= c_B^T x_B + c_N^T x_N \end{aligned} \tag{A8}$$

By submitting Eq.A7 into Eq.A8, we can get

$$P = c_B^T B^{-1} b - (c_B^T B^{-1} N - c_N^T) x_N \tag{A9}$$

Accordingly, the LP problem stated in Eq.3.1 to 3.3 can be rewritten in the following form.

Maximize:

$$P = c_B^T B^{-1} b - (c_B^T B^{-1} N - c_N^T) x_N \tag{A10}$$

Subject to

$$x_B = B^{-1}b - B^{-1}Nx_N \tag{A11}$$

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} \geq 0 \tag{A12}$$

As already described, the optimal solution which maximizes the objective function is located at one of the vertices which are constructed by the intersection of a hyperplane spanned by an objective function and a polyhedron spanned by constraints. These vertices correspond at least to basic solutions obtained by setting (n – m) nonbasic variables equal to zero in the constraints. Accordingly, all basic solutions have the possibility to be optimal. Therefore the thing to do is to obtain the conditions for basic solutions to be feasible and optimal.

(i) Conditions for Feasibility of Basic Solutions. Nonbasic variables are set to be zero in Eq.A12 in order to get basic solutions. Using Eq.A12, the condition for the basic solution to be feasible is as follows.

$$x_B = B^{-1}b \geq 0 \tag{A13}$$

(ii) Conditions for Optimality of Basic Solutions. Coefficient vector $(c_B^T B^{-1}N - c_N^T)$ in the objective function A10 is called the vector of reduced casts. According to Eq.A10, the reduced costs must be positive in order that basic solutions are optimal. Otherwise, the objective function will increase as nonbasic variables become positive.

Therefore, the condition for the basic solution to be optimal is as follows.

$$c_B^T B^{-1}N - c_N^T \geq 0 \tag{A14}$$

Eq.A13 and A14 are basic conditions in the Simplex method as well. The Simplex method gives in a step-by-step manner the optimal solution which satisfies the condition for optimality while renewing by a pivoting operation a combination of basic variables which satisfies the condition for feasibility.

As is already obvious in the proposed method, specific numerical values are not given to *A*, *b* and *c* to get the general solutions. Therefore, it is possible to obtain the condition for feasibility, the condition for optimality, and the general solutions by deriving the inverse matrix of *B* when a combination of basic variables composing a solution is arbitrarily specified. Namely, a relationship among *A*, *b* and *c* satisfying Eq.A13 and A14 is derived by assuming possible combinations of basic variables, whereas the method based on the Simplex method obtains basic solutions satisfying Eq.A13 and A14 making use of pivoting conditions in the Simplex tableau.

B. Basic Procedure for Analytical Solution

As the next step, a basic procedure to reach to the analytical solutions is clearly shown below.

Step 1. Obtain all sets of combinations of basic variables.

Step 2. Derive the inverse matrix of B for all sets of combinations of basic variables.

Step 3. Derive basic solutions, the conditions for feasibility and the conditions for optimality using Eq.A13 and A14 as follows:

$x_B = B^{-1}b$ for basic solutions;

$B^{-1}b \geq 0$ for feasibility;

$c_B^T B^{-1} N - c_N^T \geq 0$ for optimality;

Step 4. Arrange the conditions for feasibility with respect to an element b_j of b and define feasible regions for each basic solution as follows.

$$B^{-1}b = \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1m} \\ \vdots & & \vdots & & \vdots \\ b_{i11} & \cdots & b_{ij} & \cdots & b_{im} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{nm} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_j \\ \vdots \\ b_m \end{pmatrix} \geq 0$$

$$b_j \geq \underbrace{(b_{i1}b_1 + \cdots + b_{ij-1}b_{j-1} + b_{ij+1}b_{j+1} + \cdots + b_{im}b_m)}_{q_i} / b_{ij} \geq -q_i / b_{ij}$$

$$\max(-q_i / b_{ij} \,|\, b_{ij} > 0) \leq b_j \leq \min(q_i / b_{ij} \,|\, b_{ij} < 0) \quad (i = 1 \ldots m)$$

$\max(-q_i / b_{ij})$ and $\min(q_i / b_{ij})$ are termed as extreme points off the feasible region.

Step 5. Rearrange the feasible regions for all basic solutions with increasing order of magnitude of extreme points the left to the right. Rearranged extreme points on a sequence are called cut-points hereinafter.

Step 6. Line up all basic solutions based on the feasible region on the cut-point sequence. One of the solutions in a feasible region may be an optimal solution and the remaining solutions correspond to alternative feasible solutions.

Step 7. Assume one of the basic solutions to be optimal in the feasible region on the cutpoint sequence from the left to the right; proceed one-by-one in sequence to the next region and obtain the optimal solution space as the result of successive assumption of optimality.