Claims – Based Asset Pricing: Theory and Application

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This paper outlines a claims-based theory of asset pricing and estimates claim values from options data. It also interprets the estimates with new asset pricing measures based on using the state price density as a reference portfolio. In contrast to the literature's claim that distributions of claim values are much the same irrespective of estimation techniques, our measures of the market price of risk and the risks of individual claims help to distinguish sharply among different distributions. The measures suggest that in anticipation of a market downturn, the market price of risk declines while claims' measured risks increase.

Claims – Based Asset Pricing: Theory and Application

1. Introduction

This paper develops a claims-based theory of asset pricing, estimates claim values from options data, and uses the estimates to provide new interpretations of changing market conditions. A claims-based theory of asset pricing is not new, but we make new use of the state price density as a reference portfolio. In addition, we give new economic interpretations of the reference portfolio, and show how changes in market prices can be interpreted in terms of the market price of risk and individual claims' risk measures. In contrast to the literature's claim that distributions of claim values are much the same irrespective of estimation techniques, our measures help to distinguish sharply among different distributions.

Following the literature, the paper first derives a claims-based security market line (CBML) based on a linear transformation of the state price density, henceforth denoted as *L*. Since the derivation assumes only a complete market and the absence of arbitrage opportunities, its assumptions are less restrictive than those used to derive the CAPM security market line. The CAPM implicitly assumes market completeness and an absence of arbitrage opportunities, but it also explicitly assumes either that investor utility is quadratic or that returns are jointly normally distributed.

We argue below that the claims-based theory offers more advantages than disadvantages, but it also has one limitation to which the CAPM is not subject. The optimally chosen distribution of investor wealth is a linear transformation of *L* under the assumptions of the CAPM, as is the market portfolio. However, if the assumptions of the CAPM are not invoked, optimal investor choices need not take the form of a linear transformation of *L*. Thus depending on the assumptions underlying its derivation, the reference portfolio on which the CBML is based may or may not represent optimal investor choice.

Regardless of whether or not *L* represents an optimal choice, it serves as a useful reference portfolio in any arbitrage free setting because it reveals new information about market pricing processes. Like the CAPM, the claims-based theory explains asset prices in terms of a market price of risk and a measure of risk. Unlike the CAPM, the claims-based theory provides information about risk measures for unit contingent claims. In contrast to risk measures for individual securities, the measures for unit claims represent a more fundamental theoretical construct. Since claims-based measures give information about how market prices differ among claims on different states of the world, as a reference portfolio *L* is more informative than the market portfolio.

Since securities payoffs can be interpreted as packages of unit claims, the claims-based approach provides new interpretations of the factors determining their prices. First, linear combinations of claims-based risk measures define the risk measures for arbitrarily chosen securities. Second, a security with a low excess return can be regarded as being composed of proportionately more claims on states with low rates of return than can a security with a high excess rate of return. For example, if standard industrial stocks have relatively high positive excess returns while high technology stocks offer negative excess rates of returns the latter can be interpreted as offering packages of claims more heavily weighted toward those states of the world for which the unit claims have negative excess rates of return.

Empirical estimates for the claims-based theory are no more difficult to obtain than are empirical estimates for the CAPM. It is about as easy to estimate the prices and objective probabilities of claims, and consequently to determine L, as it is to determine the market portfolio. Moreover, L can be estimated at successive points in time to help interpret changes in market sentiment. In particular, our data suggest that as the market anticipates a price drop, the market price of risk declines but the risks themselves increase. Our estimates also suggest that when market prices are expected to increase the market price of risk increases and measured risks decrease.

1.1. Theoretical literature

Duffie (1996) and Pliska (1997) show how the CBML can be derived when a linear transformation of L is used as a reference portfolio. Both Duffie and Pliska assume an absence of arbitrage opportunities, but Duffie's derivation is based on regression arguments, while Pliska's employs valuation calculations. Pliska also shows that the CAPM security market line (hereafter the SML) employs a linear transformation of L as a reference portfolio, thus demonstrating that when its assumptions are invoked, the CAPM can be interpreted as a special case of claims-based asset pricing theory.

Banz and Miller (1978) advocate developing the claims-based theory of asset pricing further, but to date the only efforts of this type are the Duffie and Pliska works just mentioned. Yet the claims-based theory still merits further consideration, not least because of objections to using either quadratic utility or normality of returns assumptions. On the other hand, further development of the claims-based theory also demands additional justification. As Pliska (1997:50) observes, one of the challenges in developing a claims-based theory is to show that L is as useful a reference portfolio as is the CAPM's market portfolio. The theoretical and interpretive arguments in the rest of this paper address Pliska's challenge, and the paper's empirical sections show how the theory can be operationalized.

1.2 Empirical literature

The two main approaches to estimating claim values are based on Banz and Miller (1978) and Breeden and Litzenberger (1978). (Given the riskless rate, estimating claim values is equivalent to estimating the risk-neutral probability measure.) The Banz-Miller approach assumes that asset returns are distributed lognormally and that the Black-Scholes formula correctly reflects options' market prices. Under these assumptions the data needed to implement the Banz-Miller technique are asset price volatility and the riskless rate of interest. State definitions must also be chosen, and consistent with the assumption of lognormality, Banz-Miller define states in terms of return increments ranging from –100% to plus infinity. (This entire range of return increments must be considered in order to obtain risk neutral probabilities that sum to unity.) However, option data indicate that the underlying asset prices do not satisfy the lognormality assumption (Longstaff 1995). Similarly, other option data indicate that market consensus values do not normally extend over the range specified by Banz and Miller (Ross, 2000).

The Breeden-Litzenberger technique assumes discrete asset price distributions, and can thus employ state definitions that closely reflect properties of the data being used. Ross (2000) shows that in comparison to the Banz-Miller technique, the Breeden-Litzenberger estimates are more sensitive to parameter change, and are therefore more suitable for tracing changes in claims' prices. Among other advantages, the Breeden-Litzenberger technique permits individual claim values to be constrained explicitly. In contrast the Banz-Miller constraints are implicitly defined by the choice of lognormal distribution.

Several subsequent papers have examined topics closely related to the Banz-Miller and Breeden-Litzenberger works. Longstaff (1995) assumes an absence of arbitrage opportunities to study relations between options prices and the prices of their underlying assets. He finds that the Black-Scholes model has strong bid-ask spread, trading volume, and open interest biases, and argues that option pricing models perform significantly better if the martingale restriction is relaxed. But as already mentioned, Longstaff's study rejects strongly the assumption that the underlying asset price process is distributed lognormally. Thus an alternative to Longstaff's approach could be to relax his distributional assumption, as we do when estimating claim values with the Breeden-Litzenberger method.

Madan and Milne (1995) develop a factor-based approach to estimating claim values. Their work both requires intensive computation and can be interpreted as a generalization of the Breeden-Litzenberger method aimed at uncovering the market factors driving state prices. Bahra (1997) examines techniques for estimating risk-neutral probabilities from option prices. Bahra's preferred method assumes the implied risk-neutral density can be estimated using a weighted sum of two lognormal distributions, thus more closely approximating the actual data than can a single lognormal distribution.

Jackwerth (1999) reviews three methods for recovering implied risk-neutral probability distributions from option prices.

> "First, find a candidate risk neutral distribution, and fit the resulting option prices to the observed prices. Second, fit a function of option prices across strike prices through the observed option prices. Then use the Breeden and Litzenberger (1978) model to obtain the risk-neutral distribution. Third, fit a function of implied volatilities across strike prices through the observed implied volatilities. Next,

> calculate the function of option prices across strike prices and, finally, obtain the risk-neutral distribution." (1999:67)

Jackwerth further observes that "although there are numerous (sic) methods for recovering the risk-neutral distributions, the results tend to be rather similar unless we have very few option prices." In fact, "one can use just about any reasonable method and the resulting risk-neutral distributions will be rather similar," mainly because if sets of, say, ten to fifteen options are used the collective set of no-arbitrage bounds is so tight … that the price of an additional option … is determined to within the bid-ask spread." (1999:71). Jackwerth notes the distributions of claim values appear similar. However this paper's measures of the market price of risk and the risks of individual claims can distinguish sharply among apparently similar distributions, as shown further in the paper's fifth section.

1.3 Organization of the paper

The rest of the paper is organized as follows. Section 2, following Pliska, derives the CBML when a linear transformation of *L* is used as a reference portfolio. Section 3, also following Pliska, nests the Capital Asset Pricing Theory within the claims-based theory. Section 4 interprets *L* as a reference portfolio, and derives a particular form of CBML. Section 5 estimates the values of contingent claims and interprets the results. Section 6 concludes.

2. A Claims-Based Market Line

Consider a portfolio whose time 1 payoffs are described by a linear transformation of the state price density; $A \equiv a + bL;\ b \neq 0$. Assume further that *A* is attainable: i.e., there is a some trading strategy, that generates $V_A = a + bL$, where V_A is the time 1 value of the portfolio, and v_A its time zero value. Since

$$V_A = v_A\,(1+R_A) = a + bL, \tag{2.1}$$

it follows immediately that R_A is also a linear transformation of the state price density.

Next, consider an arbitrary portfolio with return R_X and observe that

$$E_Q\,R_X = r,$$

where E_Q means expectation under the risk neutral probability measure *Q* (cf. Pliska 1997). The last line can be rewritten

$$r = E_Q\,R_X = E(R_X\,L) = cov(R_X\,,L)+E(R_X)E(L).$$

Since $E[L]=\Sigma_k(q_k/p_k)p_k = 1$ where q_k and p_k are respectively the risk neutral and objective probabilities on a state by state basis; $k = 1, \ldots , K$,

$$r = cov(R_X ,L)+E(R_X). \tag{2.2}$$

Moreover,

$$cov(R_X, L) = (v_A / b)\, cov(R_X ,R_A).$$

and hence

$$E(R_X) – r = - (v_A / b)cov(R_X ,R_A). \tag{2.3}$$

Now consider the special case in which $X \equiv A$, so that $R_X = R_A$ and (2.3) becomes

$$E(R_A) – r = - (v_A / b)var(R_A). \tag{2.4}$$

Combining (2.3) and (2.4) yields:

$$[E(R_X) – r] /[E(R_A) – r] = cov(R_X, R_A) / var(R_A), \tag{2.5}$$

from which

$$[E(R_X) – r] = [cov(R_X, R_A) / var(R_A)][E(R_A) – r]$$

$$\equiv \beta_A[E(R_A) – r]. \tag{2.6}$$

Equation (2.6), the CBML, is similar to the SML except that the reference portfolio is $A = a + bL$ rather than the market portfolio.

If investor utility is quadratic, wealth can be expressed as an affine function of the state price density, and in this case the reference portfolio will be both an affine function of the state price density and also the solution to a quadratic optimization problem. That is, under the assumptions of the CAPM, the market portfolio can be written as $a + bL$ when a and b are chosen appropriately. The next section provides this derivation.

Under more general conditions, it will not always be possible to express the utility maximizing portfolio in the form $a + bL$. However as Section 4 shows, even in these cases $a + bL$, or more particularly L, can be interpreted as an economically meaningful choice of reference portfolio. The resulting specialized form of CBML describes excess return relationships between securities for any market in which there are no arbitrage opportunities. Thus the specialized form of CBML relates to the SML in much that same way that Ross' Arbitrage Pricing Theory relates to the Capital Asset Pricing Theory.

3. The SML as a Special Case of the CBML

This section follows Pliska (1997) to show that the classic mean-variance portfolio problem yields an optimal wealth and optimal return that are both affine functions of the state price density, meaning that the SML can be derived from the CBML when investor utility is quadratic. Moreover, in this case the reference portfolio, the market portfolio, is the solution to the investor's problem of maximizing the expected utility of a quadratic function of terminal wealth.

3.1 Alternative formulations

To obtain the desired result, this section restates the expected utility maximization problem in a form suitable for using the risk neutral computational approach described in Pliska. Assume:

1) there are no arbitrage opportunities
2) the riskless interest rate r is deterministic
3) there is an attainable portfolio with $E[R] = r$.

Pliska first shows that the classic mean-variance problem can be expressed either in terms of rates of return or investor wealth. Consider first the problem

$$min\ var(R)$$

subject to

$$E(R) = \rho \tag{3.1}$$

where R is portfolio return, and ρ is a specified scalar. Writing $V_0 = v$ and noting that $V_1 = v(1+R)$, it follows immediately that

$$min\ var(V_1)$$

subject to

$$E(V_1) = v(1+\rho)$$

can be rewritten as

$$min\ var(v(1+R))$$

subject to

$$E(v(1+R)) = v(1+\rho) \tag{3.2}$$

Consideration of the optimality conditions for (3.1) and (3.2) shows them to be unaffected by the constant v. Hence (3.2) has the same solution as (3.1). Henceforth we consider problems expressed in terms of invested wealth.

Problem (3.2) is consistent with taking a utility of the form $E(V_1) - (\gamma / 2) var(V_1)$ and solving

$$max \{ E(V_1) - (\gamma / 2)\sigma^2(V_1) \},$$

subject to

$$E(V_1) = v(1+\rho). \tag{3.3}$$

3.2 Using the risk neutral approach

For computations using the risk-neutral method, we wish to work with the function

$$max\ E[-(1/2)V_1^2 + \beta V_1]$$

subject to

$$E_Q[V_1/(1+r)] = v. \tag{3.4}$$

We are able to use (3.4) in place of (3.3) because for certain parameter choices the two problems will have the same solution. This result can be seen heuristically by writing the differentials for the isoquants of the two objective functions as respectively

$$\sigma(V_1) = (1/\gamma)(dE/d\sigma)$$

and

$$\sigma(V_1) = [\beta - E(V_1)]\ (dE/d\sigma).$$

Since the function $(dE/d\sigma)$ is defined by the efficient frontier of risky and riskless securities, it must be the same for both problems, and then if $(1/\gamma) = [\beta - E(V_1)]$, the two problems have the same solution. A more formal demonstration due to Pliska (1997: 37-40) is given below.

Next, introducing the state price density to problem (3.4) gives:

$$max \{E[-(1/2)V_1^2 + \beta V_1] - \lambda E[(L\ V_1/(1+r)) - v]$$

$$= max\ E\{-(1/2)V_1^2 + \beta V_1 - \lambda[(L\ V_1/(1+r)) - v]\ \}.$$

The first order condition for each $\omega \varepsilon \Omega$ is

$$-V_1 + \beta = \lambda L/(1+r);$$

$$V_1 = \beta - \lambda L/(1+r). \tag{3.5}$$

Substitute (3.5) into the income constraint in problem (3.3) to find λ:

$$E_Q[(\beta - \lambda L/(1+r)) / (1+r)] = v,$$

which can be simplified to

$$[(\beta - v(1+r)] / E_Q L = \lambda /(1+r). \tag{3.6}$$

3.3 Deriving the SML

Then substituting (3.6) into (3.5) and simplifying,

$$V_1^* = [\beta(E_Q L - L) / E_Q L] + [v(1+r) L/ E_Q L]. \tag{3.7}$$

Since $E(L) = 1$,

$$E_Q[L] = E(L^2) = 1 + \sigma^2(L) \geq 1,$$

and the equality is strict whenever $Q \neq P$, as we assume henceforth. The optimal solution to problem (3.4) will satisfy problem (3.2) only if

$$E[V_1^*] = v (1+\rho),$$

where V_1^* is defined in (3.7). Simplifying the last expression gives

$$E[V_1^*] = [\beta(E_Q L - 1) / E_Q L] +v(1+r)/ E_Q L = v (1+\rho).$$

Expressing the last line in terms of β gives

$$\beta = v[(1+\rho)E_Q L - (1 + r)]/ [E_Q L - 1]. \tag{3.8}$$

Finally, substituting (3.8) into (3.7),

$$V_1^* = \left[\frac{v[(1+\rho)E_Q L - (1+r)]}{E_Q L - 1}\right]\left[\frac{(E_Q L - L)}{E_Q L}\right] + \frac{v(1+r)}{E_Q L} L \tag{3.9}$$

Equation (3.9) shows that the solution of the mean-variance problem (3.1) is an affine function of the state price density. Moreover, the optimal return is:

$$R^* = (V_1^* / v) - 1 = \frac{\rho E_Q L - r}{E_Q L - 1} - \frac{[\rho - r]}{(E_Q L - 1)} L \,, \tag{3.10}$$

and (3.10) shows that R^* is also an affine function of L. Accordingly, R^* can be used for R' in (2.6), and since R^* is the solution to (3.1), the replacement yields the SML. This completes the demonstration that the SML is a special case of the CBML when investor preferences are quadratic.

If investor utility is quadratic, wealth can be expressed as a linear function of the state price density, and in this case the market portfolio will be both a linear transformation of L and also the solution to a quadratic optimization problem (Pliska 1997). That is, under the assumptions of the CAPM, the market portfolio can be written as $A = a + bL$ for appropriate choices of a and b. More specifically, recall that the SML is:

$$E[R_X] - r = \beta_M (E[R_M] - r) \tag{3.11}$$

where $\beta_M \equiv cov(R_X, R_M)/Var(R_M)$, R_M refers to the return on the market portfolio and R_X to the return to an arbitrary security. For particular values of a and b, say a_0 and b_0, the market portfolio $M = a_0 + b_0L \equiv A$, from which it follows immediately that (3.11) can be expressed in the same form as (6):

$$E[R_X] - r = \beta_A (E[R_A] - r). \tag{3.12}$$

Pliska observes that "the general version is not particularly useful unless you can identify (the reference portfolio) with an economically meaningful portfolio." This paper contends that the CBML provides useful information about the implications of arbitrage-free prices, even if the reference portfolio does not represent an optimal choice. First, we show below that the CBML relates market excess returns on securities to the covariance between their returns and the returns on L. Second, we show that L reflects market preferences relative to objective probabilities (cf. also Neave and Johnson 2000). Finally, we show that estimating L is no more difficult than estimating the market portfolio for the CAPM.

Commenting on this relationship, Pliska observes that "it is almost a lucky accident" that under the assumptions of the CAPM the market portfolio turns out to be an affine function of the state price density. "With most utility functions the return of the optimal portfolio will not be an affine function of the state price density and thus cannot play the role of R′ in result (2.6)."

> "For example, … log utility investors will want to choose portfolios whose time t=1 values are proportional to the *inverse* of the state price density. In this case the time t = 1 wealth cannot … be expressed as an affine function of the state price density. (Then) the security market line result will not hold with respect to

> any portfolio in which the log utility investor would desire to invest his or her money" (Pliska 1997, pp.50).

On the other hand, "(r)esult (2.6) is more fundamental in the sense that it applies to any single period securities market provided its general hypotheses are satisfied." Moreover, even when the reference portfolio does not represent an optimally chosen portfolio the CBML still provides useful information about the implications of arbitrage-free prices, just as the Arbitrage Pricing Theory of Ross provides an alternative to the CAPM. First, the CBML shows how market excess returns on securities are related to the covariance of their returns with the returns to *L*. Neave and Johnson (2000) develop economic interpretations of *L*, noting in particular that *L* reflects market preferences relative to objective probabilities. Duffie (1996) observes that *Q* measures the opportunity cost of wealth in different states of the world.

4. The State Price Density as Reference Portfolio

We now consider *L* itself as the reference portfolio. Note first that the time 0 market value of *L* is:

$$v_L = E_Q[L]/(1+r) = E(L^2)/(1+r) = [1+\sigma^2(L)]/(1+r), \tag{4.1}$$

since $E(L) = 1$. Hence $E[R_L]$, the expected return on *L*, is

$$E[R_L] = (r-\sigma^2(L)) / [1+ \sigma^2(L)]$$

and the expected excess return on *L* is

$$E[R_L] - r = -\sigma^2(L)(1+r)/[1+ \sigma^2(L)] \leq 0. \tag{4.2}$$

The non-positive excess return follows from the fact that the value of a unit claim represents the opportunity cost of wealth in a particular state (cf. Duffie, 1996).

Note that when the riskless rate is given, $\sigma^2(L)$ is a sufficient statistic for both v_L and $E[R_L]$. In a risk neutral market $Q \equiv P$, where *P* is the objective probability measure, and $\sigma^2(L) \equiv 0$. However if $Q \neq P$, $\sigma^2(L) > 0$ and $E[R_L] < r$. If the market is not risk neutral, some components of *L* will be greater than unity (reflecting discount states) while others will be less than unity (reflecting premium states). A claim that pays off \$1.00 in state *k* and zero in every other state has a value

$$v_k = E_Q[1_k(\omega)] = q_k / (1+r);\ \ k = 1, \dots, K, \tag{4.3}$$

where $1_k(\omega_k) = 1;\ 1_k(\omega_j) = 0;\ j \neq k$ Moreover, the time 1 certainty value of the claim, relative to its probability, is

$$v_k(1+r) / p_k = q_k / p_k = L(\omega_k);\ \ k = 1, \dots , K.$$

Finally,

$$L(\omega_k) < 1 \Leftrightarrow q_k < p_k.\ \ k = 1, \dots , K. \tag{4.4}$$

Thus, if $L(\omega_k) < 1$ the time 1 certainty value of the claim is less than the probability of the state's obtaining, while if $L(\omega_k) > 1$ the time 1 value exceeds the objective probability. Finally, even in a strictly risk-averse (or strictly risk-loving) market there may be one or more values of k such that $L(\omega_k) = 1$, in which case the claim's market-required return equals the riskless rate.

We now specialize the CBML derivation of Section 2 by using L itself as a reference portfolio. First, recall that (6) expresses the excess return on an arbitrary security as

$$[E(R_X) - r] = [cov(R_X, R_A) / var(R_A)] [E(R_A) - r]$$

$$\equiv \beta_A[E(R_A) - r].$$

where R_A is the return on a portfolio $A = a + bL$ and R_X the return on an arbitrary security. Setting $a = 0, b = 1$ gives L as the reference portfolio, in which case (6) can be rewritten as

$$E(R_X) - r = \equiv \beta_L[E(R_L) - r]. \tag{4.5}$$

If the market is not risk neutral, $[E(R_L) - r] < 0$. It follows that if $[E(R_X) - r] > 0$ then necessarily $\beta_L < 0$. If an asset has a positive expected excess return, its returns are negatively correlated with the returns on L. Note finally that (4.3) describes the excess expected return on any arbitrary portfolio, any arbitrary security, and any individual contingent claim.

Banz and Miller (1978: 656, n. 2) interpret claim values as follows:

> Each v_k can be regarded as the product of a probability and a scarcity factor. The higher the probability of a given future state, *ceteris paribus*, the higher the value is a current claim on that state. And, for equal probabilities, the current price of a claim to funds in a state in which funds are hard to come by (as in a depression) will be higher than one in an ebullient state where almost everything is paying off handsomely. Thus, a project with most of its payoff contingent on a boom will have a lower value per dollar of expected return than one whose payoffs are in the other direction.

To illustrate the effects of scarcity on the value of claim, consider the following example. The state of the economy next period will be either "boom", (ω_1), or "bust", (ω_2) with equal objective probability. There are no other possible states. Let *I* be a proxy for the state of the economy and assume that: $I_0 = 5$, $I(\omega_1) = 15$, and $I(\omega_2) = 1$. Also assume that the riskless rate, r, is 10%. Then the state prices are given by the solution to

$$5 = q_1(15/1.1) + q_2(1/1.1)$$

$$1 = q_1 + q_2;$$

for which $Q = (0.3214, 0.6780)$, and $L = (0.6428, 1.3560)$. The unit claim on state ω_1 has a value of $v_1 = 0.3214/1.1 = 0.2922$, while the claim on state ω_2 has a value of $v_2 = 0.6786/1.1 = 0.6169$. Thus, the asset that pays a unit amount in an economic depression has a larger value than an asset that pays a unit amount in an economic boom.

These results are consistent with an equilibrium approach to asset pricing.[11] With neoclassical preferences, utility is a strictly increasing strictly concave function of wealth, and the marginal utility of wealth is greater in the economic depression state, ω_2, than in the economic boom state, ω_1. Accordingly, the price of the asset that pays off in the depression state will be greater than one that pays off in the boom state.

The CBML for this example is obtained from the following data.

$$v_L = (1 + \sigma^2(L)) = (1.1276)/1.10 = 1.0251;$$

$$E(R_L) – r = -0.1245,\ \beta_1 = -4.9127;\ \beta_1 = 2.3270.$$

Hence the two claims respectively offer the excess returns:

$$E(R_1 - r) = (-4.9127)\times-0.1245 = 0.6116;$$

$$E(R_2 - r) = (2.3270)\times-0.1245 = -0.2897.$$

[11] See Pliska (1997: pp. 33-36, pp. 40-46) and Smith (1999: pp. 29-34) for more details on equilibrium pricing of financial assets.

5. Estimating Values of State Claims

Ross (2000) considers two methods of estimating state prices. The first, used by Banz and Miller[1] (1978), is based on the Black-Scholes valuation formulate. The second, due to Breeden and Litzenberger (1978), is based on a discrete state space approach. In either case option payoffs are interpreted as portfolios of contingent claims. State prices are then derived by assuming an absence of arbitrage opportunities and using the values of a sufficiently large number of call options (cf Ross 1976, Banz and Miller 1978, Breeden and Litzenberger (1978)).

5.1 Choice of Method

Ross (2000) shows that under the Garman-Banz-Miller method, the distribution of claim values depends heavily on the way in which states are defined. Sensitivity analysis shows that:

1. when constant absolute increments are used to define the state boundaries, the result is an exponentially decreasing distribution of claim prices; and
2. when constant relative increments are used to define the state boundaries the result is a normal distribution of claim prices.

Which of the above claim value distributions emerge depends on the manner in which states are defined over the non-negative real line. Our findings accord with the claim prices reported by Banz and Miller, but the latter do not explain that the pattern of claim prices is affected by the assumption of lognormality. Volatility change has little effect on the claim values. The main effect is to increase claim values for states describing lower rates of return, and to decrease the values of the claims in the higher states (Ross, 2000). Finally, since the Banz-Miller estimates are not sensitive to parameter change, we do not make further use of this technique here. Rather, we discuss the Breeden-Litzenberger approach, which is both more sensitive to parameter change and avoids making specific assumptions regarding the empirical distribution of index values.

This paper employs the Breeden and Litzenberger (1978) technique to estimate the values of contingent claims. We derive state prices by assuming an absence of arbitrage opportunities and using the values of a sufficiently large number of call options (cf Ross 1976, Banz and Miller 1978, Breeden and Litzenberger (1978)). The data we use are values for call options written against futures contracts (i.e. futures options; cf. Duffie (1988)) on the S&P 500 Index. The data, downloaded from the Chicago Mercantile Exchange (CME) website, cover the period January 1, 2000 to May 31, 2000. The details of some 26000 transactions are reported in the database. (The CBOE reports current prices for ordinary European Options written against the S&P 500 Index, but the CME offers better historical data availability.)

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[1] Banz-Miller actually use two formulae to calculate state prices. We choose the formula which Banz-Miller attribute to Garman (1976).

The definition of state boundaries and option payoffs affect the distribution of state estimates. The manner in which the method incorporates market data allows the values of state claims to change from month to month. As a consequence of the lognormality assumption, the Banz-Miller formula assigns more value to extreme states than does the Breeden-Litzenberger approach. Indeed, it can be argued that Banz-Miller overstates the values of extreme states, while the Breeden-Litzenberger approach, by avoiding the need to define extreme states, may actually understate their values. The Breeden-Litzenberger procedure is sensitive to market data, in that the estimates change substantially from month to month and appear to reflect (i) the range of relevant states; and (ii) the volatility of the underlying index. The Breeden-Litzenberger estimates are also more sensitive to changes in volatility than the Banz-Miller estimates.

5.2 Data

The data we use are values for call options written against futures contracts[2] on the S&P 500 Index.[3] The data, downloaded from the Chicago Mercantile Exchange (CME) website, cover the period January 1, 2000 to May 31, 2000. The details of some 26000 transactions are reported in the data base. We estimate state prices on options with a time to expiry of approximately 20 business days. In this paper we report four valuations obtained with the Breeden-Litzenberger technique.[4] Black (1976) shows that the value of futures options is given by a variant of the Black-Scholes formula with the exercise price being adjusted by a discount factor to reflect the valuation differences between ordinary options and the futures options presently of concern.[5]

The observed prices we use are settlement prices. Estimates of futures price volatility are taken from the 2000 Equity Index Information Guide, available at www.cme.com. Annual volatilities range from 7.80% (reported in 1995) to 20.34% (reported in 1998). To ensure appropriate state definitions, the futures prices are compared to the S&P 500 Index data, available from the Yahoo Finance website, www.finance.yahoo.com. The riskless interest rate used is a representative value for one month US treasury bills obtained from[6] the Federal Reserve Bank of New York website, “Short-Term Interest Rates.”

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[2]These instruments are sometimes called futures options; see Duffie (1989).

[3] In the ordinary options traded, for example, on the Chicago Board Options Exchange, the S&P 500 itself is the underlying asset upon which the call option prices are derived.

[4] Ross (2000) obtains state prices using both the Banz-Miller and Breeden-Litzenberger approaches. We report only the latter method because, as Ross shows, the former assigns values to states that the data indicate are regarded as unattainable.

[5] The reason for using the futures options traded on the Chicago Mercantile Exchange is ready data availability.

[6] This simplification is not of significance because sensitivity analysis shows that interest rate changes have very little impact on claim values.

All of the options studied are American calls that expire during the third week of the delivery month. Theoretically, there is nothing to be gained by early exercise of an American call written against a single security that pays no dividends; under these conditions the American and European options have the same value.[19] Although the stocks comprising the S&P 500 Index pay dividends at the rate of about 2% per month, the index itself pays no dividend, and we interpret the index options as being written against a single underlying security. As a result, we treat the futures options as if they were European instruments.

The database supports the last assumption, since the options are almost never exercised prior to maturity. In five months of S&P 500 Futures call option trading at the Chicago Mercantile Exchange there were more than 10,000 call option transactions, but fewer than *twenty* exercises actually occurred. Indeed, apart from some exercises on the last day of a maturing contract, the only early exercises for the period January 1, 2000 to May 31 2000 occurred in March. In this case early exercise occurred on only one day, but involved two options with different exercise prices.

The S&P 500 Index option contract is valued in terms of 100 times the index level. If an investor buys a call option with a strike price of 1400 and the index is at 1425 on the exercise date, then upon exercising the option, the investor receives $2,500 – one hundred times the difference between the market value of the index and the exercise price.[21]

5.3 Procedure

The time period January 1, 2000 to May 31, 2000 is divided into four-week intervals according to the CME delivery dates for one month call options written against a futures position on the S&P 500 Index. Since complete data for January was not available, January calls were omitted from further study. Our first estimate of state prices uses call prices observed on January 24, for instruments (February calls) that expire February 18. The remaining periods are February 22 to March 17 for the March calls, March 27 to April 20 for the April calls, and April 24 to May 19 for the May calls. Henceforth we refer to these periods as January-February, February-March, March-April, and April-May.

To obtain discrete-time, discrete-state estimates from the CME data, we use exercise prices for traded options with a positive open interest and a positive market value. Since the Breeden-Litzenberger method does not assume lognormality, we can define our states in accord with the data but our estimates of the state prices still need to be constrained for consistency with the underlying theory. Our procedure involves first finding the unconstrained vector of state prices, then finding a constrained vector that is nearest to the

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[19] Jarrow & Turnbull (1997, pp. 73-74)

[21] See Johnson (2000, pp. 144).

unconstrained vector in the sense of minimizing the sum of squared differences between the individual components of the two vectors. (All existing applications of the Breeden-Litzenberger technique give point estimates of claim values; further theoretical and empirical research is needed to determine statistical confidence limits on the estimates.)

To obtain discrete-time, discrete-state estimates from the CME data, we use exercise prices for traded options with a positive open interest and a positive market value. Since the Breeden-Litzenberger method does not assume lognormality, it is not necessary to supplement the state definitions with fictional contracts. However, estimates of the state prices do need to be constrained for consistency with theoretical considerations. Our procedure involves first finding the unconstrained vector of state prices, then finding a constrained vector that is nearest to the unconstrained vector in the sense of minimizing the sum of squared differences between the individual components of the two vectors. The following example outlines the method.

Suppose the index futures price can take on the values $\omega(1)=1025$, $\omega(2)=1175$, $\omega(3)=1325$, $\omega(4)=1475$, and $\omega(5)=1625$ at time T. Also, suppose that there are options with exercise prices of: E=1000, E=1150. E=1300, E=1450, and E=1600, and all expire at time T. This gives the following payoff matrix:

Call Option Payoffs at time T

Index Values	**C(1000,T)**	**C(1150,T)**	**C(1300,T)**	**C(1450,T)**	**C(1600,T)**
1025	25	0	0	0	0
1175	175	25	0	0	0
1325	325	175	25	0	0
1475	475	325	175	25	0
1625	625	475	325	175	25

Define the Table's cell values as a time T call option payoff matrix **Z.** Let **P** be a standardized payoff matrix; i.e., **P** =**Z**/25. An example of the matrix **P** is shown next.

1	0	0	0	0
7	1	0	0	0
13	7	1	0	0
19	13	7	1	0
25	19	13	7	1

Let the column vector of call market prices be **M*.** Assuming an absence of arbitrage opportunities, **PV = M*,** where **V** is a column vector representing the value of the underlying contingent claims making up the payoff to each option. Then the vector of contingent claim values can be obtained from $\mathbf{V} = \mathbf{P}^{-1}\mathbf{M}^*$. That is, $\mathbf{P}^{-1}$ can be interpreted as the weights attached to the contingent claims making up the payoffs to the call options.
If **(M*)′** is as shown next,

12.6331	6.7121	1.1045	0.0800	0.0100

then $\mathbf{V'} = (\mathbf{P}^{-1}\,\mathbf{M^*})'$ is given by

0.0100	0.0607	0.9045	0.0100	0.0100

The sum of the time t=0 claim values is 0.9952. Because a portfolio of all the contingent claims yields a deterministic payoff of 1 at time T, its time zero value must be 1/(1+r), where r is the riskless rate for the period of time between time 0, the date the market prices are observed, and time T, the options' common expiry date. In this case 0.9952 is the time t=0 present value of 1 delivered at time t=T when the riskless rate of return is r = 0.00482; i.e., when the annual interest rate is approximately $1.00482^{365/30} - 1$.

Unconstrained estimates of state prices may not sum to the present value of $1.00, they may not all be positive, or both. For these reasons we constrain the estimates as described next. Observe the following additional notation:

$\mathbf{M} \equiv \mathbf{M}^* + \boldsymbol{\varepsilon}$, a k-vector of adjusted option prices;
$\mathbf{C} \equiv$ a row vector (of length k) of adjusted claim values
$c_i \equiv$ the i^{th} element in the row vector $\mathbf{C}$

and solve the problem:

$$\min \Sigma^k_{i=1}(\varepsilon_i)^2$$

subject to

$$(\mathbf{M}\,\mathbf{P_i}^{-1}) \equiv c_i > 0, \ \ \forall\, i = 1,\ldots,k$$

$$\Sigma^k_{i=1}\, c_i = 1/(1+r)$$

where $\mathbf{P_i}^{-1}$ is the i^{th} column of $\mathbf{P}^{-1}$ and the minimization is calculated by varying the ε_i. The solution to this problem is the vector of adjusted claim values, **C**. All elements in the vector **C** are constrained to be positive and must sum to the present value of $1.00 discounted at the riskless rate.

5.4 Interpretation

During January-February the S&P Index fell from about 1400 to 1350, climbed from 1350 to 1500 during February-March, fell from 1500 to 1400 during March-April, and fluctuated around 1400 during April-May. On the basis of the realized values of the index, March-April appears to have been the period of greatest uncertainty during our observation period. Table 1 shows our claim estimates for the four successive periods. Since the estimates are made using data available at the beginning of each such period, the claim estimates reflect market anticipations over the next four weeks, for each of the periods mentioned. The state definition refers to the multiplicative effect on invested capital over the period. For example, 0.80-0.89 means the realized state differs from the original state by between 0.8000 and 0.8999 of the index's original value; 1.10-1.19 means the realized state lies between 1.1000 and 1.1999 of the index's original value, and so on.

Table 1 States and claim estimates
January-February through April-May, 2000

States	Jan-Feb.	Feb-Mar	Mar-Apr	Apr-May
0.80-0.89	0.0100	0.0100	0.2660	0.0100
0.90-0.99	0.0100	0.0607	0.4761	0.7229
1.00-1.09	0.8971	0.9045	0.1761	0.2422
1.10-1.19	0.0680	0.0100	0.0671	0.0100
1.20-1.29	0.0100	0.0100	0.0100	0.0100

Note: claim values of 0.0100 are constrained values.

Source: Adapted from Ross (2000)

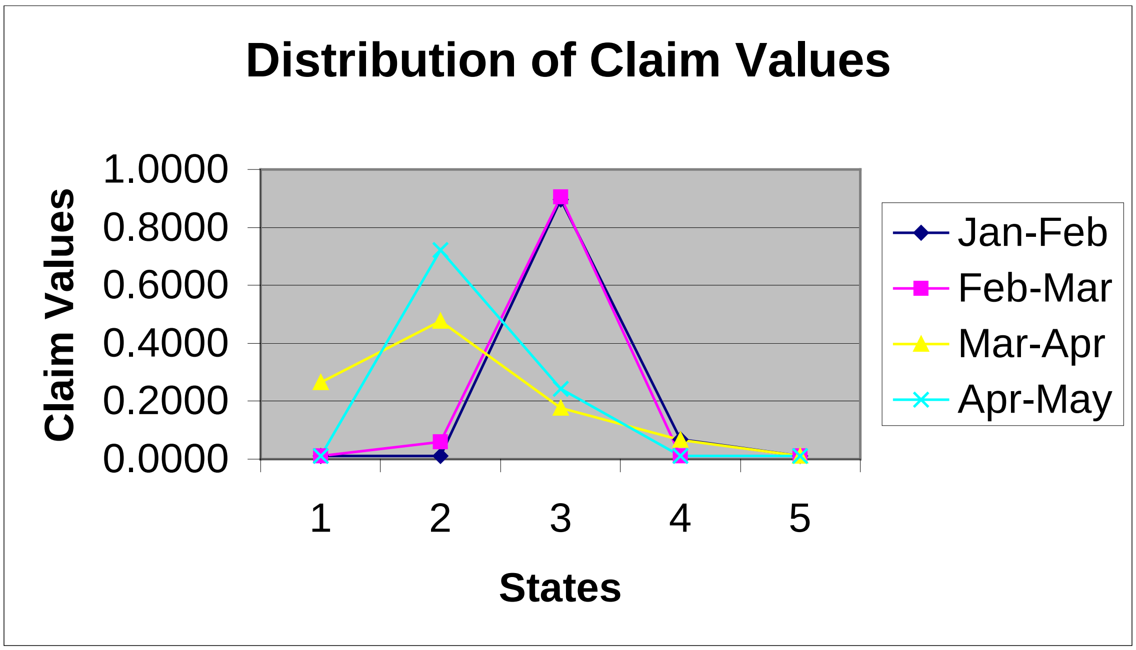

Table 1 and the accompanying graph both indicate that market consensus estimates shifted downward between the first and second pairs of periods we consider. January-February and February-March have almost identical distributions of claim values, March-April exhibits the greatest differences from the first two, and April-May represents the beginnings of a return to the January-February and February-March conditions. This interpretation is supported by the variances of the distributions, calculated assuming the states have equal objective probability in all four months. The variances are greatest and almost identical in February-March, smallest in March-April, and increase toward the February-March values in April-May. As the market revises its claim valuations downward, the valuations also become more concentrated about the mean.

The parameters of the CBML reported in Table 2 give further support to the above interpretations. First, Table 2 shows that the market price of risk; i.e., $-(E(R_L) - r)$, declined sharply in March-April; i.e., after the first two periods (January-February and February-March). The market price of risk also shows some recovery in April-May. Second, the range of beta estimates (the sign is changed for conformity with the CAPM) is much greater in Mar-April than in January-February or February-March. (The maximum values reported for $-\beta_L$ refer to claims whose estimated values exceed 0.0001; i.e., for claims whose estimated values are not affected by non-negativity constraints.) The range of the betas remains relatively low in April-May. However, since excess returns are the product of beta and the market price of risk, the excess returns do not reach the same negative values in March-April as in the other months. Finally, the ranges of beta values are consistent with the notion that, one month before the April option expired, the market pricing of claims anticipated a sharp downturn.

Table 2 CBML Estimated Parameters

	Market Price of Risk	max value, $(-\beta_L)^1$	min value, $(-\beta_L)$	Min Excess Return
Jan-Feb	.7590	2.5510	-1.0301	-.7819
Feb-Mar	.7627	3.0026	-1.0275	-.7837
Mar-Apr	.4071	4.8540	-1.4363	-.5847
Apr-May	.6626	-0.2708	-1.0990	-.7281

[1.] The reported values do not include betas calculated for claims whose values lie on the constraint of 0.0001.

6. Conclusions and Directions for Further Research

This paper develops a claims-based theory of asset pricing, estimates claim values from options data, and constructs a claims-based market line using the estimates. It then uses the CBML to obtain the market price of risk and the measures of risk, $-\beta_L$, in each of the time periods considered. The measures are then used to interpret changes in market prices. Essentially, we find that the market price of risk declines during a downturn, but measured risks increase. These effects both appear to be reversed by a recovery. We regard these findings as a promising way of helping to interpret changes in market sentiment.

The claims-based theory also offers a number of other applications possibilities. For example, after estimates of claim values have been obtained, they can be used both to price financial instruments and to value investment projects.

The development and empirical validation of a claims-based theory is still in its early stages. As one example, further research is needed to refine our empirical techniques for estimating state prices. Additional experience is needed to find the most suitable forms of state definitions for tracking changes in market sentiment. As a second example, a comparison of the constrained and unconstrained estimates indicates that market pricing does not fully remove all arbitrage possibilities, even in very active markets. The reasons for this may be that the data for different options are not observed simultaneously, but since this possibility requires individual transactions data over short periods of time, it remains to be assessed. As a third example, preliminary work with different indices suggests that claim values behave differently in different markets; in particular claim values estimated using the NASDAQ 100 Index differ from those estimated using the S&P 500 Index in much the same way as this paper's claim estimates show differences over time. The reasons for these inter-market differences also remain to be investigated further.

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1.3 Notation

The following notation is used throughout the paper[7]:

- Initial date is t = 0 and terminal date is t = 1, with trading and consumption available at these two dates.
- There is a finite sample space Ω with $K < \infty$ elements:

$$\Omega \equiv \{\omega_1, \dots , \omega_K\}.$$

Each $\omega \in \Omega$ represents a possible state of the world. There are k such states, indexed by $k \in \{ 1, 2, \dots , K \}$. At time 0 only the set of possible states is known; a particular state k will be realized at time 1. Time 1 security payoffs are dependent on which state is realized.

- A bank account process $B = \{ B_t : t = 0, 1\}$, where $B_0 = 1$ and B_1 can be either random or deterministic. At time t = 1 $B_1(\omega) \geq 1$ for all $\omega \in \Omega$. Thus, B_1 represents the time t =1 value of the bank account when $1 is deposited at time t = 0 and

$$R \equiv B_1 - 1 \geq 0,$$

where R represents the (possibly stochastic) interest rate. If the interest rate is deterministic it is denoted $r \equiv R(\omega)$ for all $\omega \in \Omega$.

- A price process $\mathbf{S} = \{S_t : t = 0,1\}$, where $S_t = (S_1(t), S_2(t),\dots,S_N(t))$, $N < \infty$, and $S_n(t)$ is the time t price of security n. These N securities are risky assets. The time 0 prices are positive scalars that are known to the investors, and at time 0 the values $S_n(1)$ are known only as non-negative random variables. At realization of $S_n(1)$, dependent on the realization of ω, becomes known to investors at time 1.

- Define a discounted price process $\mathbf{S^* = \{} S^*_t : t = 0,1 \mathbf{\}}$, where $S^*_t \equiv (S^*_1(t), S^*_2(t),\dots, S^*_N(t)$, and $S^*_n (t) \equiv S_n(t)/ B_t$, n = 1,2,…,N; t = 0,1.

- $H = (H_0, H_1,\dots, H_N)$ is a trading strategy that describes an investor's portfolio as carried forward from time t = 0 to time t = 1.

- Define a probability measure **P** such that $P(\omega) > 0$ for all $\omega \in \Omega$. Let p_k be the probability that state ω_k occurs.

- Define a risk neutral probability measure **Q** such that:

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[7] The notation is adapted from Pliska (1997).

(i) $Q(\omega) > 0, \ \forall \ \omega \in \Omega$, and
(ii) $E_Q[\Delta S^*_n] = 0$, n = 1,2,…,N.

- Let $Q(\omega_k)$ be denoted by q_k.

- Define a unit contingent claim (i.e. Arrow-Debreu security) by $1_k \equiv 1_k(\omega_k)$, where $1_k(\omega_k) \equiv 1$ and $1_k(\omega_j) \equiv 0$ for all $j \neq k$. Denote the time t = 0 value of $1_k(\omega_k)$ by v_k; $k = 1, \ldots, K$.

- Define the state price density as $L(\omega) \equiv Q(\omega) / P(\omega)$.

- Let v be the initial value of wealth at time t = 0.

- Let $V_0 = v$ be the time t = 0 value of a portfolio.

- Let V_1 be the time t = 1 value of a portfolio.